Random Graphs
Exercise Sheet 3

Question 1. Let X be a sum of indicator random variables $X = \sum_{A \in \mathcal{A}} \mathbb{1}_A$. Show that

$$\mathbb{E}((X)_k) = \sum_{A_1,\ldots,A_k \in \mathcal{A} \text{ distinct}} \mathbb{P}(\bigcap_{i=1}^{k} A_i).$$

Question 2. Suppose $p = \frac{\lambda}{n}$ show that

$$\text{Bin}(n,p) \xrightarrow{d} \text{Po}(\lambda).$$

Question 3. Let $p = \frac{c}{n}$ for $c > 0$ fixed. Determine the limit of $\mathbb{P}(K_3 \subseteq G_{n,p})$ as $n \to \infty$.

Question 4. Determine the threshold for having an isolated vertex (i.e a vertex of degree 0).

Is this threshold sharp?

(* What about a threshold for $\delta(G) \geq k$ for fixed k?)

Question 5. Let $c > 0$ be fixed and let p and μ satisfy

$$e^{-\mu} = \frac{c}{n} \quad \text{and} \quad \binom{n-1}{2} p^3 = \mu.$$

Show that

$$\lim_{n\to\infty} \mathbb{P}(\text{Every vertex in } G_{n,p} \text{ lies in a triangle}) = e^{-c}.$$

(Hint: Show that the number of vertices not lying in a triangle tends to a $\text{Po}(\mu)$ distribution)

Question 6. Let $p = \frac{c}{n}$ with $c < 1$. Show that with high probability there is a component of $G_{n,p}$ which is a tree of size $\Omega(\log n)$.